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THE STABILITY AND STABILIZATION OF THE EQUILIBRIUM POSITIONS OF NON-HOLONOMIC SYSTEMS*

A.YA. KRASINSKII

The stability and the possibility of stabilizing in the linear approximation the equilibrium positions of non-holonomic systems are studied. The number of roots of the characteristic equation on the imaginary axis in the neighbourhood of the equilibrium is greater than the number of non-integrable connections. Some of these roots may be pure imaginary. The state of the system is described by Routh variables /1, 2/. The control forces are assumed to be dependent on the momenta, the Lagrange coordinates, and their velocities.

Questions of the stability and stabilization of the stationary motions of mechanical systems were considered in /3-8/. The stabilizing forces were applied with respect to the cyclical coordinates. This method is developed in the present paper. The coordinates corresponding to momenta are not in general assumed to be cyclical and are not ignored. Sufficient conditions for stability are obtained by means of Lyapunov /9/, Malkin /10/, and Kamenkov /11/ theorems on stability in singular (essentially singular) cases. By using Routh variables, the initial structure of the forces can be preserved.

1. We consider a scleronomous non-holonomic system whose position is defined by the generalized coordinates $q_1, \ldots, q_n$, while the generalized velocities $q_1^{\cdot}, \ldots, q_n^{\cdot}$ are connected by the m non-integrable relations

$$q_\mu^{\cdot} = B_{\mu\rho}(q) q_\rho^{\cdot} \tag{1.1}$$

Here and throughout, $i, j, k = 1, 2, \ldots, n$; $\rho, s = 1, 2, \ldots, n-m$; $\mu, \sigma = n-m+1, \ldots, n$; $\xi, \eta = 1, 2, \ldots, l$; $\varkappa, \delta = l+1, \ldots, n-m$. Summation is performed over twice repeated subscripts.

Let $T^{(1)} = {}^1/_2 a_{is}^{(1)}(q) q_i^{\cdot} q_j^{\cdot}$ be the kinetic energy, and $\Pi(q)$ the potential energy, of the system. We assume that not only potential forces but also non-potential forces $Y_\rho(q, q^{\cdot})$, referred to the coordinates q_ρ, act on the system. We also assume that, in an open domain of phase space, the coefficients in the expression for the kinetic energy $a_{ij}^{(1)}(q)$, the coefficients in the equations of the connections $B_{\mu\rho}(q)$, and the potential energy $\Pi(q)$, are at least twice continuously differentiable with respect to q_i, and the generalized forces $Y_\rho(q, q^{\cdot})$ are continuous differentiable with respect to q_i, $q_i^{\cdot}$, the kinetic energy being a positive definite function of the velocities.

We shall study the stability of the equilibrium positions of a non-holonomic system with connections (1.1) with respect to all coordinates and independent velocities. Though this problem has often been studied /12/, most results to the stability of equilibrium positions in the neighbourhood of which only roots of the characteristic equation of the first approximation of the system lie on the imaginary axis. If, apart from the m zero roots, all the remaining roots lie in the left half-plane, we have the singular case of m zero roots /13/. Reduction to a singular case with a given disposition of the roots is then possible whatever the dependence of the equations of the disturbed motion on the critical variables.

In the problems treated here, when there are more roots of the characteristic equation on the imaginary axis than there are constraints, the use of reduction theory is much more difficult. First, the replacement of /13/ is no longer sufficient to determine to which variables or combinations of them the roots of the characteristic equation with zero real parts correspond, i.e., in general, to isolate the critical variables, certain linear transformations of the variables are needed. Second, both singular and non-singular critical cases are possible in such problems. It is thus more difficult to analyse the dependence of the equations of the disturbed motion on the critical variables.

Sufficient conditions for stability and instability have been obtained in some of these more difficult problems on the stability of equilibrium positions**(**A.Ya. Krasinskii, On the influence of the structure of forces on the stability of the equilibrium states of non-holonomic systems in some critical cases, Tashkent, Dep. at VINITI 30.07.80, 27.07-80. 1979.).

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However, the use of Lagrange variables to describe the states of a system led to laborious transformations, which in general change the initial structure of the forces even at the stage of isolating the critical variables. In this connection it is extremely useful to employ Routh variables, which often enable us to obtain directly the so-called critical form /10, 11/ of the equations of the disturbed motion, due to the fact that the equations for the Hamiltonian variables are solvable for the derivatives.

To describe the motion of the system, we use below the Voronets equations in Routh variables. To write these equations, we introduce the vectors and matrices

$$\begin{aligned}
&q' = (q_1, \ldots, q_n),\ r' = (q_1, \ldots, q_{n-m}),\ s' = (q_{n-m+1}, \ldots, q_n)\\
&\alpha' = (q_1, \ldots, q_l),\ \beta' = (q_{l+1}, \ldots, q_{n-m}),\ A_1 = \| a_{\xi\eta}^{(1)}(q) \|\\
&A_{12} = \| a_{\xi\varkappa}^{(1)}(q) \|,\ A_{13} = \| a_{\xi\mu}^{(1)}(q) \|,\ A_2 = \| a_{\varkappa\delta}^{(1)}(q) \|\\
&A_{23} = \| a_{\varkappa\mu}^{(1)}(q) \|,\ A_3 = \| a_{\mu\sigma}^{(1)}(q) \|,\ A_{\omega\varepsilon}' = A_{\varepsilon\omega}\\
&Y^{(1)} = (Y_1(q, q^{\cdot}), \ldots, Y_l(q, q^{\cdot})),\quad Y^{(2)} = (Y_{l+1}(q, q^{\cdot}), \ldots\\
&\qquad Y_{n-m}(q, q^{\cdot}))\\
&B_\alpha = \| B_{\mu\xi}(q) \|,\ B_\beta = \| B_{\mu\varkappa}(q) \|,\quad a(q) = \begin{Vmatrix} a_1 & a_{12} \\ a_{21} & a_2 \end{Vmatrix}\\
&a_1 = A_1 + A_{13}B_\alpha + B_\alpha' A_{31} + B_\alpha' A_3 B_\alpha',\ a_{21} = a_{12}'\\
&a_{12} = A_{12} + A_{13}B_\beta + B_\alpha' A_{32} + B_\alpha' A_3 B_\beta\\
&a_2 = A_2 + A_{23}B_\beta + B_\beta' A_{32} + B_\beta' A_3 B_\beta
\end{aligned}$$

(the prime denotes transposition). The division of the vector r into vectors α and β in a specific system depends on the nature of the dependence on these variables of the potential energy, the generalized forces, the matrix of coefficients in the equations of the constraints, and possibly also of the matrix of coefficients of the kinetic energy. Since the equations are at once written in the general case, the division of r into vectors α and β is as yet arbitrary.

We introduce the momenta $p = \partial T/\partial \beta^{\cdot}$, where $T = 1/2 r^{\cdot\prime} a(q) r^{\cdot}$ is the kinetic energy of the system, expressed in terms of the independent velocities. We write the Voronets equations in Routh variables in the vector-matrix form

$$\begin{aligned}
&\alpha^{\cdot} = \alpha_1,\quad a^* \alpha_1^{\cdot} = -\Pi_\alpha - \gamma p^{\cdot} - \alpha_1' [a_{(\alpha)}^* + a_{(\beta)}^* \gamma + a_{(s)}^* (B_\alpha - B_\beta \gamma') +\\
&\quad {}^1/_2 a_{[\alpha]}^* + {}^1/_2 a_{[s]}^* B_\alpha] \alpha_1 - \alpha_1' [a_{(\beta)}^* b_2 + a_{(s)}^* B_\beta b_2 + \gamma_{(\alpha)} - \gamma_{(\beta)} \gamma +\\
&\quad \gamma_{(s)} (B_\alpha - B_\beta \gamma') - \gamma_{[\alpha]} - \gamma_{[s]} B_\alpha] p - p' [\gamma_{(\beta)} b_2 + \gamma_{(s)} B_\beta b_2 +\\
&\quad {}^1/_2 b_{2(\alpha)} + {}^1/_2 b_{2(\alpha)} + {}^1/_2 b_{2(s)} B_\alpha] p + \theta' \Omega_1 + Y^{(1)}(q, \alpha_1, p) - B_\alpha \Pi_s\\
&\beta^{\cdot} = -\gamma' \alpha_1 + b_2 p,\quad p^{\cdot} = \alpha_1' ({}^1/_2 a_{[\beta]}^* + {}^1/_2 a_{[s]}^* B_\beta) \alpha_1 +\\
&\quad \alpha_1' (\gamma_{[\beta]} + \gamma_{[s]} B_\beta) p - {}^1/_2 p' (b_{2[\beta]} + b_{2[s]} B_\beta) p - \Pi_\beta - B_\beta \Pi_s + \theta' \Omega_2 +\\
&\quad Y^{(2)}(q, \alpha_1, p),\quad s^{\cdot} = (B_\alpha - B_\beta \gamma') \alpha_1 + B_\beta b_2 p
\end{aligned} \tag{1.2}$$

Here,

$$\begin{aligned}
&a^* = a_1(q) - a_{12}(q)\, b_2(q)\, a_{21}(q),\ b_2 = a_2^{-1}(q)\\
&\gamma = a_{12}(q) b_2(q),\quad \theta = [A_{31} + A_3 B_\alpha - (A_{32} + A_3 B_\beta) \gamma'] \alpha_1 +\\
&\quad (A_{32} + A_3 B_\beta) b_2 p,\quad \Omega_\nu = \Big[\frac{\partial B_\nu}{\partial \alpha} - \frac{\partial B_\alpha}{\partial q_\nu} + \frac{\partial B_\nu}{\partial s} B_\alpha -\\
&\quad \left(\frac{\partial B_\nu}{\partial \beta} - \frac{\partial B_\beta}{\partial q_\nu} + \frac{\partial B_\nu}{\partial \zeta} B_\beta \right) \gamma' - B_\nu (B_{\alpha[s]} - B_{\beta[s]} \gamma') \Big] \alpha_1 +\\
&\quad \left(\frac{\partial B_\nu}{\partial \beta} - \frac{\partial B_\beta}{\partial q_\nu} + \frac{\partial B_\nu}{\partial s} B_\beta - B_\nu B_{\beta[s]} \right) b_2 p,\quad \Omega_1 = \| \Omega_\xi \|\\
&\Omega_2 = \| \Omega_\delta \|,\quad \Pi_\alpha = \frac{\partial \Pi}{\partial \alpha},\quad \Pi_\beta = \frac{\partial \Pi}{\partial \beta},\quad \Pi_s = \frac{\partial \Pi}{\partial s},
\end{aligned}$$

B_ν is the ν-th column of the matrix $B(q)$, and $W_{(q)}$ and $W_{[q]}$ denote respectively for any matrix $W(q) = \| w_{ij}(q) \|$ the "vectors" with matrix components $\| \partial w_{i\nu}/\partial q_k \|, \| \partial w_{ij}/\partial q_\nu \|$, where ν is the number of the "vector" component.

2. We consider the stability of a point of the manifold of equilibrium postions

$$\frac{\partial \Pi}{\partial r} + B'(q) \frac{\partial \Pi}{\partial s} + Y(q, 0, 0) = 0 \tag{2.1}$$

We assume here without loss of generality /12/ that the equilibrium position under investigation is the origin

$$q = 0 \tag{2.2}$$

For any matrix $W(q)$ we introduce the representation

$$W(q) = W^\circ + W'^1, \quad W^\circ = W(0).$$

Here and below, the superscript after the asterisk denotes the order of the lowest terms in the expansion of the relevant expression. In Eqs.(1.2) we make the replacement of /13/, which in our notation takes the form

$$s = z + B_\alpha{}^\circ\alpha + B_\beta{}^\circ\beta \tag{2.3}$$

and we isolate the linear approximation. We have

$$\begin{aligned}
&\alpha^{\cdot} = \alpha_1,\ a^{*\circ}\alpha_1{}^{\cdot} = -\gamma^\circ p^{\cdot} - (C_1{}^* + P_1)\,\alpha - P_{12}\beta - Z_1 z + Q_1 + \\
&\quad R,\ \beta^{\cdot} = -\gamma^{\circ\prime}\alpha_1 + b_2{}^\circ\rho + N \\
&p^{\cdot} = -P_{21}\alpha - (C_2{}^* + P_2)\,\beta - Z_2 z + Q_2 + K \\
&z^{\cdot} = [(B_\alpha{}'^1 - B_\beta{}'^1 b_2)\,a_{21} - B_\beta{}^\circ b_2{}'^1 a_{21}]\,\alpha_1 + (B_\beta b_2{}'^1 + B_\beta{}'^1 b_2{}^\circ)\ p
\end{aligned} \tag{2.4}$$

Here, Q_1, Q_2 are the linear terms in the expansions of the vectors $Y^{(1)}(q, \alpha_1, p)$, $Y^{(2)}(q, \alpha_1, p)$ respectively, while the meaning of the other notation is as follows:

$$\begin{aligned}
&R = M + a^{*0}b'^1[-\gamma^\circ p^{\cdot} - (C_1{}^* + P_1)\,\alpha - P_{12}\beta - Z_1 z + Q_1 + M \\
&M = -(a_{12}{}'^1 b_2 + a_{12}{}^\circ b_2{}'^1)\,p - \Pi_\alpha{}'^2 - B_\alpha \Pi_s{}'^2 - [C_{31}\alpha + C_{32}\beta + \\
&\quad C_3(z + B_\alpha{}^\circ\alpha + B_\beta{}^\circ\beta)]\,[B_{\alpha(\alpha)}\alpha + B_{\alpha(\beta)}\beta + B_{\alpha(s)}(z + B_\alpha{}^\circ\alpha + B_\beta{}^\circ\beta)] + \\
&\quad Y^{(1)\prime 2} - {}^1\!/_2\alpha_1{}'\overset{*}{a}_{(\alpha)}\alpha_1 - (-\gamma'\alpha_1 + b_2 p)'\,\overset{*}{a}_{(\beta)}\alpha_1 - \\
&\quad [(B_\alpha - B_\beta\gamma')\,\alpha_1 + b_2 p]'\,\gamma_{(s)}p - {}^1\!/_2\alpha_1{}'b_{2(\alpha)}p + \theta'\Omega_1 + \\
&\quad {}^1\!/_2\alpha_1{}'\gamma_{(s)}B_\alpha p + {}^1\!/_2\alpha_1{}'\overset{*}{a}_{(s)}B_\alpha\alpha_1 - {}^1\!/_2 p' b_{2(s)}B_\alpha p \\
&b(q) = (a^*(q))^{-1}, \quad N = -(a_{12}{}'^1 b_2 + a_{12}{}^\circ b_2{}'^1)\,\alpha_1 + b_2{}'^1 p \\
&K = {}^1\!/_2\alpha_1{}'\overset{*}{a}_{(\beta)}\alpha_1 + \alpha_1{}'\overset{*}{a}_{(\beta)}p - {}^1\!/_2 p' b_{2(\beta)}p - \Pi_\beta{}'^2 - B_\beta\Pi_s{}'^2 - \\
&\quad [C_{31}\alpha + C_{32}\beta + C_3(z + B_\alpha{}^\circ\alpha + B_\beta{}^\circ\beta)]\,[B_{\beta(\alpha)}\alpha + B_{\beta(\beta)}\beta + \\
&\quad B_{\beta(s)}(z + B_\alpha{}^\circ\alpha + B_\beta{}^\circ\beta)] + Y^{(2),2} + {}^1\!/_2\alpha_1{}'\overset{*}{a}_{(s)}B_\beta\alpha_1 + \\
&\quad \alpha_1{}'\gamma_{(s)}B_\beta p - {}^1\!/_2 p' b_{2(s)}B_\beta p + \theta'\Omega_2
\end{aligned}$$

$$\begin{aligned}
&C_1{}^* + P_1 = C_1 + C_{13}B_\alpha{}^\circ + B_\alpha{}^{\circ\prime}C_{31} + B_\alpha{}^{\circ\prime}C_3 B_\alpha{}^\circ + L'(B_{\alpha(\alpha)} + B_{\alpha(s)}B_\alpha{}^\circ) \\
&P_1{}' = -P_1,\ C_1{}^{*\prime} = C_1{}^*,\ Z_1 = C_{13} + B_\alpha{}^{\circ\prime}C_3 + L'\overset{\circ}{B}_{\alpha(s)} \\
&P_{12} = C_{12} + C_{13}B_\alpha{}^\circ + B_\alpha{}^{\circ\prime}C_{32} + B_\beta{}^{\circ\prime}C_3 B_\beta{}^\circ + L'(\overset{\circ}{B}_{\alpha(s)}B_\beta{}^\circ + \overset{\circ}{B}_{\alpha(\beta)}) \\
&C_2{}^* + P_2 = C_2 + C_{23}B_\beta{}^\circ + B_\beta{}^{\circ\prime}C_{32} + B_\beta{}^{\circ\prime}C_3 B_\beta{}^\circ + \\
&\quad L'(\overset{\circ}{B}_{\beta(s)}B_\beta{}^\circ + \overset{\circ}{B}_{\beta(\beta)}) \\
&P_2{}' = -P_2, \quad C_2{}^{*\prime} = C_2{}^*, \quad Z_2 = C_{23} + B_\beta{}^{\circ\prime}C_3 + L'\overset{\circ}{B}_{\beta(s)} \\
&P_{21} = C_{21} + C_{23}B_\alpha{}^\circ + B_\beta{}^{\circ\prime}C_{31} + B_\beta{}^{\circ\prime}C_3 B_\alpha{}^\circ + L'(\overset{\circ}{B}_{\beta(s)}B_\alpha{}^\circ + \overset{\circ}{B}_{\beta(\alpha)}) \\
&C_1 = \|\overset{\circ}{\Pi}_{\cdot\xi\eta}\|,\ C_{12} = \|\overset{\circ}{\Pi}_{\cdot\xi\delta}\|,\ C_{13} = \|\overset{\circ}{\Pi}_{\cdot\xi\mu}\| \\
&C_2 = \|\overset{\circ}{\Pi}_{\cdot\varkappa\delta}\|, \quad C_{23} = \|\overset{\circ}{\Pi}_{\cdot\varkappa\mu}\|, \quad C_3 = \|\overset{\circ}{\Pi}_{\cdot\mu\sigma}\| \\
&\Pi_{\cdot ij} = \partial^2\Pi/\partial q_i\partial q_s, \quad L = \|(\partial\Pi/\partial q_\mu)^\circ\|, \quad C'_{\omega\varepsilon} = C_{\varepsilon\omega}
\end{aligned}$$

Disregarding the terms that arise in the expansion of the vectors Q_1, Q_2 in the phase variables, the characteristic equation of the first approximation of system (2.4) can be written as

$$\begin{vmatrix}
E_l\lambda & -E_l & 0 & 0 & 0 \\
C_1{}^* + P_1 & a^{*\circ}\lambda & P_{12} & \gamma^\circ\lambda & Z_1 \\
0 & \gamma^{\circ\prime} & E_t\lambda & -b_2{}^\circ & 0 \\
P_{21} & 0 & C_2{}^* + P_2 & E_t\lambda & Z_2 \\
0 & 0 & 0 & 0 & E_m\lambda
\end{vmatrix} = 0 \tag{2.5}$$

Corresponding to the variable z, we have m zero roots of this equation. On studying the remaining roots, and also, in the case of extra roots with zero real parts and the absence of roots in the right half-plane, the nature of the dependence of the equations of motion on the critical variables, we can obtain some assertions about the stability of the equilibrium positions.

Note 1. We have mentioned that, if the number of roots of Eq.(2.5) on the imaginary axis is greater than the number of non-integrable constraints, then both singular and non-singular critical cases of several zero and pure imaginary roots are possible. We will confine ourselves here to assertions when, first, the question is solved by reduction of the cases in question to singular cases, and second, when the statements can be made in a similar way to

Thomson-Thet-Chetayev theorems and can be proved by means of well-known results /14/ on the influence of the structure of forces on the stability of motion. It is therefore natural to impose conditions of the equation type, which in general will not be the necessary conditions for stability, since their appearance is due to the critical cases considered here, and to the fact that, to date, no effective sufficient conditions have been obtained for the stability of systems with linear non-potential positional forces.

3. Assume that potential forces with energy $\Pi(q)$ act on the non-holonomic system, together with non-potential generalized forces, such that $Y(q, 0) = 0$. Then, the vectors of the linear non-potential forces are $Q_1 = -Q_{11}\alpha_1 - Q_{12}p$, $Q_2 = -Q_{21}\alpha_1 - Q_{22}p$, where $Q_{11}, Q_{12}, Q_{21}\ Q_{22}$ are constant matrices of suitable dimensionalities.

We will consider some cases of instability and stability of the equilibrium positions, in the neighbourhood of which the number of zero roots of Eq.(2.5) is greater than the number m of constraints.

Assertion 1. Let the quadratic part of the potential energy be independent of the coordinates β, the constraints being such that we have

$$B_\beta^\circ = 0,\ \overset{\circ}{B}_{\alpha(\beta)} = 0,\ \overset{\circ}{B}_{\beta(\beta} = 0 \tag{3.1}$$

The equilibrium position (2.2) is unstable if

$$\det\begin{Vmatrix} C_1^* + P_1 & Q_{12} \\ P_{21} & Q_{22}\end{Vmatrix} < 0 \tag{3.2}$$

Proof. Under these conditions, $m + t$ zero roots of Eq.(2.5) correspond to the variables β, z, while the remaining roots are found from an equation whose unattached term is equal to the determinant (3.2).

Corollary 1. If linear forces in the velocities $\beta^\cdot$ as well as linear positional forces in β, are not present in the system, then the point (2.2) is unstable if

$$\det \| C_1^* + P_1 - \gamma P_{21} \| < 0 \tag{3.3}$$

Now, $m + 2t$ zero roots of the characteristic Eq.(2.4) correspond to the variables β, p, z. The determinant (3.3) is equal to the unattached term of the equation from which the remaining roots are found. We see that, for systems in which positional forces, linear in part of the coordinates, are not present, the question of stability with respect to the first approximation depends, not only on the positional generalized forces which are linear in the other coordinates, but also on the matrices of coefficients of the forces, linear in the velocities, and on the matrix of coefficients of the kinetic energy.

Assertion 2. Let the following conditions hold in the neighbourhood of the point (2.2):

a) the potential energy is independent of the coordinates β, while the velocities of these coordinates do not enter into the equations of constraints;

b) the acting forces and the matrix $B(q)$ are such that linear generalized forces are not present in the matrices P_1 and Q_{21}.

If the matrix C_1^* and the symmetric component matrices Q_{11}, Q_{22} are positive definite, then the equilibrium position (2.2) is asymptotically stable with respect to the velocities and stable with respect to the coordinates under the action of linear generalized forces with any matrices Z_1, Q_{12} and any skew-symmetric component matrices Q_{11}, Q_{22}.

Proof. Corresponding to the variables β, z we have $m + t$ zero roots of Eq.(2.5), while the real parts of the remaining roots are negative /14/. To reduce the problem to the singular case of $m + t$ zero roots, it is in general sufficient to make a Lyapunov-type replacement in the coordinates α. The system here has an $(m + t)$-parametric manifold of equilibrium positions, which, under our conditions, in asymptotically stable.

Note 2. Condition a) of Assertion 2 is satisfied, in particular, if the coordinates β are cyclical in the sense of /15, 16/. Vanishing of the matrix P_1 implies the absence of non-potential positional linear forces in the coordinates α, which is always the case if there are no terms linear in s in the potential energy, and if the vector α is one-dimensional, i.e., is a scalar.

Note 3. If the second equation of system (2.4) contains no free variables z, the replacement in coordinates α becomes unnecessary and the asymptotic stability is retained in these variables.

We can similarly obtain:

Assertion 3. Let the following conditions hold:

a) the potential energy is independent of β, while the potential forces contain no free coordinates s;

b) the velocities $\alpha^{\cdot}$ do not enter into the equations of the constraints, and furthermore, $B_\beta(q)|_{\alpha=0}=0$.

If the acting forces are such that, in the neighbourhood of point (2.2), the matrix C_1 and the symmetric component matrices Q_{11}, Q_{22} are positive definite and linear forces with matrices P_{21}, Q_{21} are not present, then the point (2.2) is asymptotically stable with respect to the velocities and the coordinates α, and stable with respect to coordinates β, s under the action of linear forces with any matrix Q_{12} and any skew-symmetric component matrices Q_{11}, Q_{22}.

Note 4. In the conditions imposed, there only remains in the matrix P_{21} the term $L'\overset{\circ}{B}_{\beta(\alpha)}$, which vanishes in particular if there are no terms linear in s in the expression for the energy $\Pi(q)$. Notice also that, in this assertion, the conditions of the Lyapunov-Malkin theorem are immediately satisfied in the initial variables.

Example. We consider an inhomogeneous sphere, mass m, radius R, whose central ellipsoid of inertia is an ellipsoid of revolution, while the centre of mass is not the same as the geometric centre of the sphere, and the axis of dynamic symmetry passes through the geometric centre of the sphere. The stability of the equilibrium of this sphere on a horizontal rough plane was considered in /17/. Here we shall study the stability of the equilibrium positions when the sphere touches and turns on a rough plane inclined at an angle δ to the horizontal. The state of the system is given by five generalized coordinates: the Cartesian coordinates x, y of the point of contact of the sphere with the plane, the Rezal angles θ, ψ, and the angle φ of rotation about the axis of dynamic symmetry. The x axis is parallel to the horizontal, and the y axis is upwards from the inclined plane.

The Lagrange function, formed without allowing for the constraints, and the equations of the constraints, are

$$\begin{gathered}L = {}^1/_2 m(x^{\cdot 2}+y^{\cdot 2}) + ml\,[x^{\cdot}(\theta^{\cdot}\sin\psi\sin\theta - \psi^{\cdot}\cos\psi\cos\theta) + \\ \theta^{\cdot}y^{\cdot}\cos\theta] + {}^1/_2(A+ml^2)(\theta^{\cdot 2}+\psi^{\cdot 2}\cos^2\theta) + {}^1/_2 C(\varphi^{\cdot}-\psi^{\cdot}\sin\theta)^2 - \\ mg\,[y\sin\delta + (R - l\cos\theta\cos\psi)\cos\delta + l\sin\theta\sin\delta] \\ x^{\cdot} = R\psi^{\cdot} - R\sin\theta\varphi^{\cdot}, \quad y^{\cdot} = -R\cos\psi\theta^{\cdot} - R\sin\psi\cos\theta\varphi^{\cdot}\end{gathered}$$

Here, l is the distance of the sphere centre from the centre of mass, A is the axial moment of inertia, C is the central equatorial moment of inertia, and g is the acceleration due to gravity. We can obviously assume that $0 \leqslant \delta < \pi/2$.

In addition to gravity forces, let the sphere be acted on by dissipative forces with the energy dissipation function /17/

$$F = {}^1/_2\,[h_1R^{-2}(x^{\cdot 2}+y^{\cdot 2}) + h_2(\varphi^{\cdot}\cos\psi\cos\theta - \theta^{\cdot}\sin\psi)^2]$$

where h_1, h_2 are respectively the coefficients of viscous friction of rolling and rotation. The manifold of equilibrium positions is given by

$$\theta^* = \arcsin(Rl^{-1}\sin\delta) - \delta, \quad \psi^* = 0 \tag{3.4}$$

since, with $\theta = \pi/2$, there is in general no equilibrium. Equilibria are possible only when $Rl^{-1}\sin\delta \leqslant 1$. Denoting by θ the deviation from θ^*, and introducing the vectors $q' = (\theta, \psi, \varphi, x, y)$, $\beta = \varphi$, $\alpha' = (\theta, \psi)$, $s' = (x, y)$, we find that the determinant (3.2) is equal to

$$\begin{gathered}m^{-1}\cos(\theta^*+\delta)\,g^2l^2\cos\theta^*(\sin\theta^* h_1 d + h_2\cos\theta^*) \\ d = \sin\theta^*\cos\delta - R\sin\delta\end{gathered} \tag{3.5}$$

By Assertion 1, the point $\theta=0, \psi=0, \varphi=0$ of manifold (3.4) is unstable if expression (3.5) is negative. The sign of this expression is influenced, not only by the values of $\cos(\theta^*+\delta)$ and $\cos\theta^*$, but also by the ratio of h_1 to h_2, since, in accordance with (3.4), d changes sign depending on δ and R/l. In particular, $d<0$ if $\delta \geqslant 0{,}1$, $l \geqslant 0{,}6R$. Then, (3.5) is negative if $h_2\cos\theta^*\cos\delta > h_1\operatorname{tg}\theta^*$. We have stability of this equilibrium position for $\cos\theta^* > 0$ with a special choice of the coefficients of viscous friction of rolling and rotation

$$h_2^* = \frac{h_1^*(Cm^{-1}\cos\theta^* + Rl\sin^2\theta^*)}{\cos\theta^*(Cm^{-1} - Rl\cos\theta^* + R^2)}$$

When there are no dissipative forces, the point is unstable, by Corollary 1 with $\cos(\theta^*+\delta)\cos\theta^* < 0$. A knowledge of the structure of matrix $a(q)$ and of the matrix of linear positional forces enables this result to be refined; instability occurs when $\cos(\theta^*+\delta) < 0$.

4. Let us now examine the stability of equilibrium positions, in the neighbourhood of which the characteristic Eq.(2.5) has pure imaginary as well as zero roots. The structure of the equations of disturbed motion is then quite different from the structure in the cases studied above: the right-hand sides of the **truncated** /10/ system now necessarily contain terms linear in the critical variables, while the vector of critical variables contains, in addition

to the coordinates, at least part of the independent velocities. As a result, the problem cannot be reduced to a singular case by a replacement of Lyapunov type in the non-critical coordinates only, without a replacement in the corresponding velocities. The replacement has to be made with respect to the entire vector of non-critical variables, and this does not in general lead to vanishing of the non-linear terms in the equations of the constraints.

Thus, remaining in the context of the theory of singular cases, we shall impose conditions such that e.g., Kamenkov's theorems on stability in the essentially singular case (/11/, Sect. 39) hold directly in the initial variables. Notice that it is now impossible to satisfy such conditions by imposing conditions only on the matrix of coefficients in the equations of the constraints and on the generalized forces; we also need conditions on the matrix of coefficients of the kinetic energy.

Let us quote an example of an assertion about the stability of the equilibrium positions of a non-holonomic system, in the neighbourhood of which Eq.(2.5) has pure imaginary roots.

Assertion 4. Let the following conditions hold:

a) the matrix of coefficients of the kinetic energy is such that A_{12}, A_{23}, $A_{2(\alpha)}$ vanish for $\alpha = 0$;

b) the matrix of coefficients in the equations of the constraints does not contain any free variables β, s, while $B_\beta(q) = 0$ for $\alpha = 0$;

c) resistance forces with complete dissipation act with respect to velocities $\alpha^\cdot$, while the matrix $C_1{}^* + P_1$ is symmetric and positive definite.

Let one of the following requirements be satisfied:

d) the potential forces contain no free coordinates β, s and only gyroscopic forces act with respect to the velocities $\beta^\cdot$, while $\det Q_{22} > 0$, i.e., t is an even number;

e) the potential forces contain no free coordinates s, while the coordinates β appear freely only in the linear forces with matrix $C_2{}^* + P_2$, while this matrix is symmetric and positive definite, and forces, linear in the velocities $\beta^\cdot$, are not present; then the equilibrium position (2.2) is asymptotically stable with respect to velocities $\alpha^\cdot$ and coordinates α, and is stable with respect to velocities $\beta^\cdot$ and coordinates β, s under the action with respect to velocities $\alpha^\cdot$ of any gyroscopic linear forces and any non-linear forces which contain no free variables β, $\beta^\cdot$, s.

The assertion follows from the stability theorem in the essentially singular case /11/ of $m + t$ zero and pure imaginary roots when condition d) holds, and of m zero and $2t$ pure imaginary roots when condition e) holds. In the former case, the non-holonomic system has, in addition to an $(m + t)$-parametric manifold of equilibrium positions, a t-parametric manifold of periodic motions; in the latter case, it has an m-parametric manifold of equilibrium positions and a $2t$-parametric manifold of periodic motions.

5. We formulate the problem of stabilizing the equilibrium positions of a non-holonomic system with constraints (1.1) by applying linear forces with respect to the coordinates β. Using our assertions about stability, we can prove the following concerning the possibilities of such stabilization.

Assertion 5. Under the conditions a), b) of Assertion 3 and the action of only potential forces, the equilibrium position (2.2) can be stabilized up to asymptotic stability in the velocities and coordinates α, and stability in the remaining coordinates, by the application with respect to coordinates β of the control forces

$$u = (-M_1 + P_{21})\,\alpha - M_2\alpha_1 - M_3 p$$

In particular, if the matrix C_1 is positive definite, this stabilization is realized by the forces

$$u = (-M_2 M_3 + P_{21})\,\alpha - M_2\alpha_1 - M_3 p \tag{5.1}$$

where M_3 is any matrix with a positive-definite symmetric component, and the symmetric component of matrix γM_2 is positive definite as a result of the choice of matrix M_2.

Proof. Under these conditions and the action of forces (5.1), the system of equations of the disturbed motion (2.4) has the form

$$\begin{aligned}
&\alpha^\cdot = \alpha_1,\ a^{*\circ}\alpha_1 = -C_1\alpha + \gamma\,(P_{21} + M_1)\,\alpha + \gamma M_2\alpha_1 + \gamma M_3 p + R \\
&\beta^\cdot = -\gamma^{\circ\prime}\alpha_1 + b_2{}^\circ p + N \\
&p^\cdot = -(P_{21} + M_1)\,\alpha - M_2\alpha_1 - M_2 p + K \\
&z^\cdot = B_\beta{}'^1 b_2 a_{21}\alpha_1 + (B_\beta b_2{}'^1 + B_\beta{}'^1 b_2{}^\circ)\,p
\end{aligned} \tag{5.2}$$

where the non-linear terms K, N, R vanish for $\alpha = 0, \alpha_1 = 0, p = 0$.

After the non-singular linear replacement $p = y - M_2\alpha$, instead of (5.2) we obtain a system for which all the conditions of the Lyapunov-Malkin theory on stability in the singular case of $m + t$ zero roots, corresponding to the variables β, z, are satisfied.

Example. Consider, in Routh variables, the problem of stabilizing the equilibrium position of Chaplygin's sledge on an inclined plane, under the action of gravity forces only. Disregarding the non-holonomic constraint and its equation, the Lagrange function is /17/

$$L = {}^1/_2 m\,[(x^{\cdot} + l\varphi^{\cdot}\cos\varphi)^2 + (y^{\cdot} + l\varphi^{\cdot}\sin\varphi)^2 + k^2\varphi^{\cdot 2}] - mg\sin\delta\,(y - l\cos\varphi), \quad y^{\cdot} = x^{\cdot}\operatorname{tg}\varphi$$

Here, m is the mass of the sledge, x, y are the Cartesian coordinates of the point of contact of the skate and the plane, φ is the angle between the line of intersection of the skate plane and the inclined plane and the x axis, l is the distance from the centre of mass to the axis perpendicular to the plane at the point of contact, k is the radius of inertia, g is the acceleration due to gravity, and δ is the angle of iclination of the plane.

We introduce the Routh variables $\alpha = \varphi,\ \beta = x,\ p = \partial L/\partial x^{\cdot}$. We study the stability of the point

$$x = 0, \quad y = 0, \quad \varphi = 0 \tag{5.3}$$

of the manifold $\sin\varphi^* = 0$ of equilibrium positions. The characteristic equation of the first approximation system of equations of disturbed sledge motion

$$\alpha^{\cdot} = \alpha_1, \quad m\,(k^2 + l^2)\,\alpha_1^{\cdot} = -(mgl\sin\delta)\,\alpha - lp^{\cdot} + R$$
$$\beta^{\cdot} = -l\alpha_1 + m^{-1}p + N, \quad p^{\cdot} = -(mg\sin\delta)\,\alpha + K$$
$$y^{\cdot} = \operatorname{tg}\alpha\,(-l\alpha_1 + m^{-1}p)$$

has five zero roots in the neighbourhood of the point (5.3). By Assertion 5, the force (5.1) (with $P_{21} = mg\sin\delta$), applied with respect to the x coordinate, stabilizes the point (5.3) up to asymptotic stability with respect to all velocities and the φ coordinate and up to stability with respect to x, y.

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